

Scaling Algebras and Renormalization Group in Algebraic Quantum Field Theory.

II. Instructive Examples

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Abstract: The concept of scaling algebra provides a novel framework for the general structural analysis and classification of the short distance properties of algebras of local observables in relativistic quantum field theory. In the present article this method is applied to the simple example of massive free field theory in $s = 1, 2$ and 3 spatial dimensions. Not quite unexpectedly, one obtains for $s = 2, 3$ in the scaling (short distance) limit the algebra of local observables in massless free field theory. The case $s = 1$ offers, however, some surprises. There the algebra of observables acquires in the scaling limit a non-trivial center and describes charged physical states satisfying Gauss' law. The latter result is of relevance for the interpretation of the Schwinger model at short distances and illustrates the conceptual and computational virtues of the method.

1 Introduction

The structure of local observables in relativistic quantum field theories at short distances is in many respects of physical interest. It is of relevance for the interpretation of physical states at small spacetime scales, the classification of the possible ultraviolet properties of local observables and the clarification of the corresponding algebraic structures. One may also hope that a general model-independent understanding of this issue will shed light on the constructive problems in local quantum physics [10].

A promising step towards the solution of these problems has been taken in [8], where the basic ideas of renormalization group theory have been adapted to the Haag-Kastler framework of relativistic quantum field theory. It was shown in this analysis that to each algebra of local observables there exists an associated scaling algebra on which renormalization group (scaling) transformations act in a canonical manner.

With the help of this device one can define and classify the scaling limit of any given theory [8], analyze the relation between phase space properties and the nature of the scaling limit [2] and introduce concepts for the description of the particle and symmetry features of a theory at small scales [3].

It is the aim of the present article to illustrate the computational aspects of this method by applying it to the simple case of the theory of a massive free scalar field in s spatial dimensions. Following is a very brief outline of our results; a more detailed summary will be given at the end of this introduction.

It turns out that, in the cases $s = 2, 3$, one obtains in the scaling limit the algebra of observables in the corresponding massless free field theories. Thus, according to the classification in [8], these theories have a unique quantum scaling limit. Thinking of the conventional field-theoretic approach to the renormalization group, this result may not be unexpected. Nevertheless it is of interest since it illustrates the basic message of [8] that for the short distance analysis one need not exhibit specific renormalization group transformations. It is sufficient to identify the observables at different scales by considering operator-functions of a scaling parameter which have a few general properties and which exist in abundance. This somewhat abstract approach has the virtue of being model independent, but it is still sufficiently concrete in order to carry out explicit computations.

In the case of free field theory in $s = 1$ spatial dimensions, where the conventional field theoretic approach to the short distance analysis is hampered by infrared problems, the method of the scaling algebra reveals its full strength. There it turns out that the algebra of observables in the scaling limit is a (central) extension of the algebra generated by the massless free field in exponentiated Weyl form. The presence of a center shows that, for $s = 1$, the vacuum states appearing in the scaling limit can be mixed, in contrast to theories in higher dimensions, where these states are always pure [8].

More interestingly, the present method allows one to exhibit in the scaling limit physical states carrying a (global) gauge charge in the sense of [5] for which Gauss' law holds. This result provides a marked illustration of the fact that the charge structure of a theory may differ substantially from that of its scaling limit. It is in particular of relevance for the interpretation of the Schwinger model where the algebra of the local (gauge-invariant) observables is known to be isomorphic to that generated by a free massive field [11]. Thus, whereas this theory does not have any charged superselection sectors at finite scales, there appear physical states in the scaling limit carrying an "electric charge". The presence of these states may be interpreted as a manifestation of "partons" in the Schwinger model, i.e. observable particle-like structures appearing at small spacetime scales which have no counterpart at large scales. For a more detailed discussion of this issue and its relation to the notion of confinement, cf. [3].

Hence, even though the models underlying our present investigation are rather trivial, the results nicely illustrate and exemplify various points in the abstract analysis carried out in [8].

For the convenience of the reader we recall in the remainder of this introduction various notions and results from [8] and establish our notation. The method of

the scaling algebra relies on the following fundamental properties of any physically acceptable theory [10].

1. (Locality) The observables of the theory generate a net of local algebras over $(1+s)$ -dimensional Minkowski space \mathbb{R}^{1+s} , i.e. an inclusion preserving map

$$\mathcal{O} \rightarrow \mathcal{A}(\mathcal{O}) \quad (1.1)$$

from the set of open double cones $\mathcal{O} \subset \mathbb{R}^{1+s}$ to unital C^* -algebras $\mathcal{A}(\mathcal{O})$. The algebra generated by all local algebras $\mathcal{A}(\mathcal{O})$ (as a C^* -inductive limit) is denoted by \mathcal{A} . The net is supposed to satisfy the principle of locality (Einstein causality), i.e. all pairs of operators which are assigned to spacelike separated double cones commute.

2. (Covariance) The Poincaré group \mathcal{P}_+^\uparrow is represented by automorphisms of the net. Thus for each $(\Lambda, x) \in \mathcal{P}_+^\uparrow$ there is an automorphism $\alpha_{\Lambda, x} \in \text{Aut } \mathcal{A}$ such that, in an obvious notation,

$$\alpha_{\Lambda, x}(\mathcal{A}(\mathcal{O})) = \mathcal{A}(\Lambda\mathcal{O} + x) \quad (1.2)$$

for any double cone \mathcal{O} . In [8] this fundamental postulate was amended by the condition that for each $A \in \mathcal{A}$ the function $(\Lambda, x) \mapsto \alpha_{\Lambda, x}(A)$ is strongly continuous. In the present analysis we require continuity only with respect to the translations.

3. (States) The physical states are described by positive, linear and normalized functionals ω on \mathcal{A} . By the GNS-construction, any state ω gives rise to a representation π_ω of \mathcal{A} on a Hilbert space \mathcal{H}_ω , and there exists a unit vector $\Omega_\omega \in \mathcal{H}_\omega$ such that

$$\omega(A) = \langle \Omega_\omega, \pi_\omega(A)\Omega_\omega \rangle, \quad a \in \mathcal{A}. \quad (1.3)$$

States describing the vacuum are distinguished by the fact that, on the corresponding Hilbert space \mathcal{H}_ω , there is a continuous unitary representation $U_\omega(\Lambda, x)$ of \mathcal{P}_+^\uparrow which leaves the unit vector Ω_ω invariant, satisfies the relativistic spectrum condition (positivity of the energy) and implements the action of \mathcal{P}_+^\uparrow on the observables,

$$U_\omega(\Lambda, x)\pi_\omega(A)U_\omega(\Lambda, x)^{-1} = \pi_\omega(\alpha_{\Lambda, x}(A)), \quad A \in \mathcal{A}. \quad (1.4)$$

Any state of physical interest is assumed to be locally normal to the vacuum state (i.e. its restriction to any local algebra can be represented by a vector in the Hilbert space of the vacuum representation, cf. [10, Sec. V.2]).

We amend these physically well-motivated assumptions by a condition of a more technical nature in order to simplify the subsequent discussions. Namely we assume that the local algebras are continuous from the outside,

$$\mathcal{A}(\mathcal{O}) = \bigcap_{\mathcal{O}_1 \supset \overline{\mathcal{O}}} \mathcal{A}(\mathcal{O}_1), \quad (1.5)$$

where $\overline{\mathcal{O}}$ denotes the closure of \mathcal{O} . If a given net $\mathcal{O} \rightarrow \mathcal{A}(\mathcal{O})$ fails to comply with this additional condition, one can always proceed to a corresponding regularized

net $\mathcal{O} \rightarrow \mathcal{A}_{\text{reg}}(\mathcal{O}) := \bigcap_{\mathcal{O}_1 \supset \overline{\mathcal{O}}} \mathcal{A}(\mathcal{O}_1)$ which has all the desired properties, as is easily checked. In view of this simple fact we may assume without essential loss of generality that all local nets appearing in our analysis are continuous in the sense of relation (1.5).

Within this setting the short distance analysis is carried out as follows. One first proceeds from the given net and automorphisms (\mathcal{A}, α) at spacetime scale $\lambda = 1$ (in appropriate units) to the corresponding nets $(\mathcal{A}_\lambda, \alpha^{(\lambda)})$ describing the theory at arbitrary scale $\lambda \in \mathbb{R}^+$. This is easily done by setting for given $\lambda > 0$

$$\mathcal{A}_\lambda(\mathcal{O}) := \mathcal{A}(\lambda\mathcal{O}), \quad \alpha_{\Lambda, \lambda x}^{(\lambda)} := \alpha_{\Lambda, \lambda x}. \quad (1.6)$$

The identification of observables at different scales can then be accomplished by considering functions \underline{A} of the scaling parameter $\lambda > 0$ whose values \underline{A}_λ are to be interpreted as observables in the nets $(\mathcal{A}_\lambda, \alpha^{(\lambda)})$, $\lambda \in \mathbb{R}^+$. Clearly, any such function \underline{A} establishes some relation between observables at different scales.

With this simple idea in mind one is led to the concept of scaling algebra $\underline{\mathcal{A}}$ which consists of certain specific functions $\underline{A} : \mathbb{R}^+ \rightarrow \mathcal{A}$ with properties described below. The algebraic operations in $\underline{\mathcal{A}}$ are pointwise defined by the corresponding operations in \mathcal{A}_λ , $\lambda \in \mathbb{R}^+$, and there is a C^* -norm on $\underline{\mathcal{A}}$ given by

$$\|\underline{A}\| := \sup_{\lambda > 0} \|\underline{A}_\lambda\|. \quad (1.7)$$

The local structure of \mathcal{A} is lifted to $\underline{\mathcal{A}}$ by setting

$$\underline{\mathcal{A}}(\mathcal{O}) := \{\underline{A} : \underline{A}_\lambda \in \mathcal{A}_\lambda(\mathcal{O}), \quad \lambda \in \mathbb{R}^+\}. \quad (1.8)$$

Hence $\mathcal{O} \rightarrow \underline{\mathcal{A}}(\mathcal{O})$ is a local net over \mathbb{R}^{1+s} and $\underline{\mathcal{A}}$ is defined as its C^* -inductive limit. One can also lift the action of the Poincaré group in the underlying theory to a corresponding action of automorphisms $\underline{\alpha}_{\Lambda, x}$ on $\underline{\mathcal{A}}$, which are given by

$$(\underline{\alpha}_{\Lambda, x}(\underline{A}))_\lambda := \alpha_{\Lambda, \lambda x}(\underline{A}_\lambda). \quad (1.9)$$

It is crucial to demand that $\underline{\mathcal{A}}$ consists only of elements on which the translations act strongly continuously, i.e. for each $\underline{A} \in \underline{\mathcal{A}}$ there holds

$$\|\underline{\alpha}_x(\underline{A}) - \underline{A}\| \rightarrow 0 \quad \text{as} \quad x \rightarrow 0. \quad (1.10)$$

Heuristically speaking, the latter constraint amounts to the condition that, for given \underline{A} , the operators \underline{A}_λ occupy, for all values of λ , certain regions in “phase space” with a fixed volume [8]. Hence, whereas the scale of spacetime changes along the graph of \underline{A} , the scale \hbar of action is kept fixed. In [8] it was assumed that also the Lorentz transformations act in a strongly continuous manner on $\underline{\mathcal{A}}$; yet in order to simplify the discussion we do not impose this stronger condition here and consider the somewhat larger scaling algebra $\underline{\mathcal{A}}$ consisting of all functions satisfying the preceding, weaker conditions. It should be emphasized that this algebra is fixed by these conditions once a local net (\mathcal{A}, α) is given.

The structure of the physical states ω in the underlying theory at small spacetime scales can now be analyzed as follows. Given ω one defines a lift of this state to the scaling algebra at scale $\lambda \in \mathbb{R}^+$ by setting

$$\underline{\omega}_\lambda(\underline{A}) := \omega(\underline{A}_\lambda), \quad \underline{A} \in \underline{\mathcal{A}}. \quad (1.11)$$

Let $(\pi_\lambda, \mathcal{H}_\lambda)$ be the GNS-representation of $\underline{\mathcal{A}}$ which is fixed by $\underline{\omega}_\lambda$. Then one considers the net

$$\mathcal{O} \rightarrow \underline{\mathcal{A}}(\mathcal{O})/\ker \pi_\lambda, \quad \underline{\alpha}_{\Lambda,x}^{(\lambda)}, \quad (1.12)$$

where \ker means “kernel” and $\underline{\alpha}_{\Lambda,x}^{(\lambda)}$ is the induced action of the Poincaré-transformations $\underline{\alpha}_{\Lambda,x}$ on this quotient. This net is isomorphic to the theory $(\mathcal{A}_\lambda, \alpha^{(\lambda)})$ at scale λ [8].

We recall in this context that two local nets $(\mathcal{A}_a, \alpha^{(a)})$ and $(\mathcal{A}_b, \alpha^{(b)})$ are said to be isomorphic if there exists an isomorphism $\phi : \mathcal{A}_a \rightarrow \mathcal{A}_b$ which preserves locality, $\phi(\mathcal{A}_a(\mathcal{O})) = \mathcal{A}_b(\mathcal{O})$ for each double cone \mathcal{O} , and intertwines the action of the Poincaré-transformations, $\phi \circ \alpha_{\Lambda,x}^{(a)} = \alpha_{\Lambda,x}^{(b)} \circ \phi$; isomorphic nets describe the same physics.

With these preparations one is led to the following canonical definition of the scaling limit of a theory. One first considers the set $SL(\omega)$ of limit points (in the weak- $*$ -topology) of the net of states $\{\underline{\omega}_\lambda\}_{\lambda>0}$ for $\lambda \rightarrow 0$. This set of states is always non-empty by standard compactness arguments. We denote the elements of $SL(\omega)$ by $\omega_{0,\iota}$, where ι is an element of some index set, and recall that $\omega_{0,\iota} \in SL(\omega)$ means that there exists some directed set \mathbb{K} (depending on ι) such that, for some net of scaling parameters λ_κ , $\kappa \in \mathbb{K}$, which converges to zero, one has

$$\lim_{\kappa} \underline{\omega}_{\lambda_\kappa}(\underline{A}) = \omega_{0,\iota}(\underline{A}), \quad \underline{A} \in \underline{\mathcal{A}}. \quad (1.13)$$

Thus, roughly speaking, the sequence of states $\underline{\omega}_\lambda$ need not converge for $\lambda \rightarrow 0$, but there exist always convergent subsequences. The following general facts about the scaling limit states $\omega_{0,\iota}$ have been established in [8].

1. $SL(\omega)$ does not depend on the chosen physical (locally normal) state ω .
2. Each $\omega_{0,\iota} \in SL(\omega)$ is a vacuum state on $\underline{\mathcal{A}}$. (In the present, more general setting one has, however, no control on the continuity properties of the Lorentz transformations.)

With this information it is clear how to define, in analogy to the case of finite scales, theories associated with the scaling limit $\lambda \rightarrow 0$. Picking $\omega_{0,\iota} \in SL(\omega)$ one proceeds to its GNS-representation $(\pi_{0,\iota}, \mathcal{H}_{0,\iota})$ and defines the net and automorphisms

$$\mathcal{O} \rightarrow \mathcal{A}_{0,\iota}(\mathcal{O}) := \underline{\mathcal{A}}(\mathcal{O})/\ker \pi_{0,\iota}, \quad \alpha_{\Lambda,x}^{(0,\iota)}, \quad (1.14)$$

where $\alpha_{\Lambda,x}^{(0,\iota)}$ denotes the induced action of $\underline{\alpha}_{\Lambda,x}$ on the quotient net. This net has the same general properties as the underlying theory (possibly apart from outer continuity; in that case we pass to the regularized net without further mentioning).

We mention as an aside that in our computations we will frequently make use of the general fact that the abstract net defined in (1.14) is isomorphic to the concrete net of C^* -algebras and automorphisms given by

$$\mathcal{O} \rightarrow \pi_{0,\iota}(\underline{\mathcal{A}}(\mathcal{O})), \quad \text{Ad } U_{0,\iota}(\Lambda, x), \quad (1.15)$$

where $U_{0,\iota}(\Lambda, x)$ are the unitaries representing the Poincaré group on $\mathcal{H}_{0,\iota}$.

Given these results, there arises the interesting problem of whether the nets $(\mathcal{A}_{0,\iota}, \alpha^{(0,\iota)})$ depend on the choice of the state $\omega_{0,\iota} \in SL(\omega)$. As was discussed in [8], there are the following possibilities.

1. All nets $(\mathcal{A}_{0,\iota}, \alpha^{(0,\iota)})$ are isomorphic to the trivial net $(\mathbb{C}1, \text{id})$, where id denotes the trivial automorphism (“classical scaling limit”).
2. All nets $(\mathcal{A}_{0,\iota}, \alpha^{(0,\iota)})$ are isomorphic and non-trivial (“unique quantum scaling limit”). If the respective isomorphisms connect also the vacuum states $\omega_{0,\iota}$, one has “a unique vacuum structure in the scaling limit”.
3. Not all of the nets $(\mathcal{A}_{0,\iota}, \alpha^{(0,\iota)})$ are isomorphic (“degenerate scaling limit”).

Which case is at hand depends of course on the underlying theory. As was argued in [8], case 2 may be expected to hold in many theories of physical interest. In spite of the existence of an abundance of scaling limit states $\omega_{0,\iota}$, which may be attributed to the fact that the scaling algebra $\underline{\mathcal{A}}$ contains the orbits of local observables under arbitrary renormalization group transformations [8], these theories have a well-defined and non-trivial scaling limit. Phrased differently, in this generic case it does not matter which renormalization group transformation one uses in order to determine the scaling limit, all transformations yield the same result.

The same remark applies to theories leading to case 1, for which the scaling limit is trivial. It is only for theories corresponding to case 3 that the short distance structure cannot be described by a single net. For a tentative physical interpretation of these cases see [8]. Examples of nets with such remarkable short distance properties will be presented elsewhere [7].

In the present investigation we will apply this scheme to the nets $(\mathcal{A}^{(m)}, \alpha^{(m)})$ generated by the free field of mass m in s spatial dimensions. These nets will be introduced in Sec. 2 as concrete operator algebras in some “standard representation” which differs from the familiar Fock representation but will be convenient for the present analysis since it accommodates the free nets of arbitrary mass in a transparent manner. In Sec. 3 we will analyze the structure of the scaling limit nets $(\mathcal{A}_{0,\iota}^{(m)}, \alpha^{(m;0,\iota)})$ in the cases $s = 2, 3$ and show that they are isomorphic to the net $(\mathcal{A}^{(0)}, \alpha^{(0)})$ generated by the free massless field and that the isomorphisms connect the respective vacuum states. Hence, in this specific sense these theories have a unique quantum scaling limit and vacuum structure in this limit.

The case $s = 1$ is treated in Sec. 4, where we show that, as already mentioned, the algebras $\mathcal{A}_{0,\iota}^{(m)}$ have a non-trivial center and hence the vacuum states $\omega_{0,\iota}^{(m)}$ are mixed. We also exhibit physical states $\omega_{q,\iota}^{(m)}$ on the nets $(\mathcal{A}_{0,\iota}^{(m)}, \alpha^{(m;0,\iota)})$ which carry some charge q and coincide with the vacuum state $\omega_{0,\iota}^{(m)}$ on the observables in the right and

left spacelike complement of some sufficiently large double cone. But they are disjoint from the vacuum on the algebra of observables of the full spacelike complement of any double cone, no matter how large. Therefore one can determine the charge of these states in the spacelike complement of any bounded region and this fact may be regarded as an algebraic version of Gauss' law. The article concludes with a list of open problems which are outlined in Sec. 5.

2 Standard representations of free fields

We deal in the present article with the theories of the free scalar field of arbitrary mass. It is therefore convenient to employ the formulation of free field theory based on the time zero fields and their canonically conjugate momenta since they do not depend on the mass.

We introduce the fields and conjugate momenta in exponentiated form by considering the unitary operators $W(f)$, where f is any element of $\mathcal{D}(\mathbb{R}^s)$, the space of complex valued test-functions with compact support in the configuration space \mathbb{R}^s . The canonical commutation relations then turn into the Weyl relations

$$W(f)W(g) = e^{-\frac{i}{2}\sigma(f,g)}W(f+g), \quad f, g \in \mathcal{D}(\mathbb{R}^s), \quad (2.1)$$

where the symplectic form σ is given by

$$\sigma(f, g) := \text{Im} \int d^s x \, \overline{f(\mathbf{x})} g(\mathbf{x}). \quad (2.2)$$

The $*$ -algebra generated by all Weyl operators is denoted by \mathfrak{W} . On \mathfrak{W} we introduce various automorphisms of geometrical significance. The action of the spatial translations \mathbb{R}^s on the Weyl operators is given by

$$\alpha_{\mathbf{x}}(W(f)) := W(\tau_{\mathbf{x}}f), \quad \mathbf{x} \in \mathbb{R}^s, \quad (2.3)$$

where $(\tau_{\mathbf{x}}f)(\mathbf{y}) := f(\mathbf{x} - \mathbf{y})$. For given mass $m \geq 0$, we define corresponding time translations by setting

$$\alpha_t^{(m)}(W(f)) := W(\tau_t^{(m)}f), \quad t \in \mathbb{R}. \quad (2.4)$$

Here, $(\tau_t^{(m)}f)(\mathbf{x})$ is the unique solution of the Klein-Gordon equation of mass m with initial data $\text{Re}f + i\text{Im}f$, $(\Delta - m^2)\text{Im}f + i\text{Re}f$. More explicitly,

$$\begin{aligned} (\tau_t^{(m)}f) &:= (\cos(t\mu_m) + i\mu_m^{-1}\sin(t\mu_m))\text{Re}f \\ &+ i(\cos(t\mu_m) + i\mu_m\sin(t\mu_m))\text{Im}f, \end{aligned} \quad (2.5)$$

where μ_m acts in momentum space according to $\widetilde{(\mu_m f)}(\mathbf{p}) := \sqrt{\mathbf{p}^2 + m^2}\tilde{f}(\mathbf{p})$. (The tilde denotes, as usual, the Fourier-transform.) Because of the propagation properties of the solutions of the Klein-Gordon equation, $(\tau_t^{(m)}f)$ has support in a ball of radius $r + |t|$ if f has support in a ball of radius r ; hence $\mathcal{D}(\mathbb{R}^s)$ is stable under the action of $\tau_t^{(m)}$.

It is apparent that the automorphisms $\alpha_{\mathbf{x}}$ and $\alpha_t^{(m)}$ commute for arbitrary $m \geq 0$. In contrast, the time translations corresponding to different values of m do not commute. In order to simplify notation we put $\alpha_x^{(m)} := \alpha_t^{(m)} \circ \alpha_{\mathbf{x}}$ and similarly $\tau_x^{(m)} := \tau_t^{(m)} \circ \tau_{\mathbf{x}}$ for $x = (t, \mathbf{x}) \in \mathbb{R}^{1+s}$.

In a similar way one can introduce a mass dependent action $\alpha_{\Lambda}^{(m)}$ of the Lorentz-transformations on \mathfrak{W} , but we dispense with giving explicit formulas. We will also need the action of length scale transformations (dilations) on \mathfrak{W} . They are fixed by setting

$$\sigma_{\lambda}(W(f)) := W(\delta_{\lambda}f), \quad \lambda > 0, \quad (2.6)$$

where

$$(\delta_{\lambda}f)(\mathbf{x}) := \lambda^{-\frac{s+1}{2}}(\text{Re}f)(\lambda^{-1}\mathbf{x}) + i\lambda^{-\frac{s-1}{2}}(\text{Im}f)(\lambda^{-1}\mathbf{x}). \quad (2.7)$$

It is straightforward to establish the following relation between the Poincaré-transformations and dilations:

$$\sigma_{\lambda} \circ \alpha_{\Lambda, x}^{(\lambda m)} = \alpha_{\Lambda, \lambda x}^{(m)} \circ \sigma_{\lambda}, \quad \lambda > 0. \quad (2.8)$$

Next we introduce the vacuum states on \mathfrak{W} corresponding to the different time evolutions. Given $m \geq 0$, we put

$$\omega^{(m)}(W(f)) := e^{-\frac{1}{2}\|f\|_m^2}, \quad f \in \mathcal{D}(\mathbb{R}^s), \quad (2.9)$$

where (excluding the singular case $s = 1, m = 0$)

$$\|f\|_m^2 := 2^{-1} \int d^s p \, |\mu_{m, \mathbf{p}}^{-1/2}(\widetilde{\text{Re}f})(\mathbf{p}) + i\mu_{m, \mathbf{p}}^{1/2}(\widetilde{\text{Im}f})(\mathbf{p})|^2 \quad (2.10)$$

and $\mu_{m, \mathbf{p}} := \sqrt{\mathbf{p}^2 + m^2}$. The extension of $\omega^{(m)}$ to \mathfrak{W} is fixed by linearity and describes the vacuum state in the theory of mass m . There holds in particular $\omega^{(m)} \circ \alpha_{\Lambda, x}^{(m)} = \omega^{(m)}$ and $\omega^{(m)} \circ \sigma_{\lambda} = \omega^{(\lambda m)}$.

The present analysis is greatly simplified by the following result due to Eckmann and Fröhlich [9]. In its formulation there enter the sub-algebras $\mathfrak{W}(G)$ of \mathfrak{W} which are assigned to the regions $G \subset \mathbb{R}^s$ and are generated by all Weyl operators $W(g)$ where g are test-functions with $\text{supp } g \subset G$.

Proposition 2.1 *Let $s = 2$ or 3 and let $G \subset \mathbb{R}^s$ be bounded. Then the restricted (partial) states $\omega^{(m)} \upharpoonright \mathfrak{W}(G)$, $m \geq 0$, are normal with respect to each other. If $s = 1$ this statement holds for $m > 0$.*

This result and the local action of the spacetime transformations allow one to describe and analyze the nets of local von Neumann algebras generated by the various free fields of different mass in a fixed “standard representation” of \mathfrak{W} which is induced by any one of the vacuum states. In the subsequent section we treat the cases $s = 2, 3$ and take as a standard state the $m = 0$ vacuum, whereas in our discussion of the case $s = 1$, we fix some vacuum state with $m > 0$. In order to simplify notation, we use

the symbol $W(f)$ also for the concrete Weyl operators in the GNS-representation of \mathfrak{W} induced by the chosen standard state.

Within the chosen standard representation we can define the net of local von Neumann algebras on Minkowski space corresponding to the theory of mass m as follows. Given any double cone $\mathcal{O}_0 \subset \mathbb{R}^{1+s}$ with base G in the time $t = 0$ plane and any Poincaré transformation Λ, x , we set

$$\mathcal{R}^{(m)}(\Lambda\mathcal{O}_0 + x) := \{\alpha_{\Lambda,x}^{(m)}(W(g)) : \text{supp } g \subset G\}'' , \quad (2.11)$$

where the prime denotes the commutant. In this way we obtain a local net $\mathcal{O} \rightarrow \mathcal{R}^{(m)}(\mathcal{O})$ of von Neumann algebras on the underlying Hilbert space which, by the result of Eckmann and Fröhlich, is isomorphic to the net generated by the free field of mass m on the Fock space corresponding to $\omega^{(m)}$. Moreover, the automorphisms $\alpha_{\Lambda,x}^{(m)}$ extend to the local von Neumann algebras $\mathcal{R}^{(m)}(\mathcal{O})$ and act covariantly on the net, i.e.

$$\alpha_{\Lambda,x}^{(m)}(\mathcal{R}^{(m)}(\mathcal{O})) = \mathcal{R}^{(m)}(\Lambda\mathcal{O} + x) . \quad (2.12)$$

Note, however, that for m different from the mass of the chosen standard state the time translations $\alpha_t^{(m)}$ are not unitarily implemented in the underlying Hilbert space. An analogous statement holds for the Lorentz boosts.

Let us briefly indicate the advantage of the present standard representation of the various local nets which is only locally normal with respect to the familiar Fock representations. What we gain is the equality of the local algebras $\mathcal{R}^{(m)}(\mathcal{O}_0)$ for arbitrary m and any double cone \mathcal{O}_0 with base in the time $t = 0$ plane, cf. relation (2.11). As a consequence there holds for arbitrary double cones \mathcal{O} and masses m_1, m_2

$$\mathcal{R}^{(m_1)}(\mathcal{O}) \subset \mathcal{R}^{(m_2)}(\mathcal{O}_0) \quad (2.13)$$

whenever \mathcal{O}_0 is some double cone with base at $t = 0$ which contains \mathcal{O} . Moreover, choosing $\omega^{(0)}$ as the standard state in the cases $s = 2, 3$, we are able to use in the analysis of the massive theories the invariance of $\omega^{(0)}$ under the dilations σ_λ , and their covariant action on the massless net,

$$\sigma_\lambda(\mathcal{R}^{(0)}(\mathcal{O})) = \mathcal{R}^{(0)}(\lambda\mathcal{O}) . \quad (2.14)$$

Since the scaling limit of a theory does not depend on the choice of a locally normal state [8], the present setting proves to be most convenient.

As was mentioned in the Introduction, the short distance analysis of a local net of von Neumann algebras requires the passage to a corresponding subnet of C^* -algebras consisting of operators which transform strongly continuously under the action of Poincaré transformations or, more generally, spacetime translations. We restrict attention here to the latter case and consider for fixed m the weakly dense subnet of $\mathcal{O} \rightarrow \mathcal{R}^{(m)}(\mathcal{O})$ given by

$$\mathcal{O} \rightarrow \mathcal{A}^{(m)}(\mathcal{O}) := \{A \in \mathcal{R}^{(m)}(\mathcal{O}) : \lim_{x \rightarrow 0} \|\alpha_x^{(m)}(A) - A\| = 0\} . \quad (2.15)$$

This net still transforms covariantly under the Poincaré transformations $\alpha_{\Lambda,x}^{(m)}$ and, in the case $m = 0$, also under dilations σ_λ . Its C^* -inductive limit is denoted by $\mathcal{A}^{(m)}$ and the various vacuum states extend to this algebra by local normality under the conditions stated in Proposition 2.1. We also note that the algebras $\mathcal{A}^{(m)}(\mathcal{O})$ are continuous from the outside,

$$\mathcal{A}^{(m)}(\mathcal{O}) = \bigcap_{\mathcal{O}_1 \supset \overline{\mathcal{O}}} \mathcal{A}^{(m)}(\mathcal{O}_1), \quad (2.16)$$

as a consequence of the outer continuity of the von Neumann algebras $\mathcal{R}^{(m)}(\mathcal{O})$. We emphasize, however, that because of the continuity requirement in (2.15) it is no longer true that the algebras $\mathcal{A}^{(m)}(\mathcal{O}_0)$ coincide for different m and fixed double cones \mathcal{O}_0 based at time $t = 0$, in contrast to their weak closures.

3 Computation of the scaling limit for $s = 2, 3$

In the present section our objective is to prove the following result which provides full information about the scaling limit theories of the free scalar fields of any mass in three- and four-dimensional Minkowski-spacetime.

Theorem 3.1 *Let $s = 2, 3$, $m \geq 0$, and let $\omega_{0,\iota}^{(m)}$ be any scaling limit state of the theory $(\mathcal{A}^{(m)}, \alpha^{(m)}, \omega^{(m)})$ of a free scalar field of mass m in $(1 + s)$ -dimensional Minkowski-spacetime. Then the associated scaling limit theory $(\mathcal{A}_{0,\iota}^{(m)}, \alpha^{(m;0,\iota)}, \omega_{0,\iota}^{(m)})$ is net-isomorphic to the theory $(\mathcal{A}^{(0)}, \alpha^{(0)}, \omega^{(0)})$ of the massless free scalar field in the same spacetime dimension, and the corresponding net-isomorphism connects $\omega_{0,\iota}^{(m)}$ and $\omega^{(0)}$.*

Remark. This result implies that, according to the classification in [8], these free field theories have a unique quantum scaling limit with a unique vacuum structure. A similar theorem holds for the scaling limit theories of the local nets if one imposes the continuity requirements (2.15) and (1.10) for the whole Poincaré group.

The proof of this result proceeds in several steps. To begin with we recall (cf. the discussion in Sec. 2) that we are working in the standard representation of \mathfrak{W} which is induced by the mass zero vacuum state $\omega^{(0)}$. Now let $\omega_{0,\iota}^{(m)}$ be a scaling limit state of $\omega^{(m)}$, so that

$$\omega_{0,\iota}^{(m)}(\underline{A}) = \lim_{\kappa} \omega_{\lambda_\kappa}^{(m)}(\underline{A}), \quad \underline{A} \in \underline{\mathcal{A}}^{(m)}, \quad (3.1)$$

for a suitable subnet λ_κ , $\kappa \in \mathbb{K}$, of positive real numbers converging to 0. We denote the GNS-representation of $\omega_{0,\iota}^{(m)}$ by $(\pi_{0,\iota}, \mathcal{H}_{0,\iota}, \Omega_{0,\iota})$. It is our aim to show that the required net-isomorphism ϕ is obtained by assigning to $\pi_{0,\iota}(\underline{A})$, $\underline{A} \in \underline{\mathcal{A}}^{(m)}(\mathcal{O})$, the operators

$$\phi(\pi_{0,\iota}(\underline{A})) := w - \lim_{\kappa} \sigma_{\lambda_\kappa}^{-1}(\underline{A}_{\lambda_\kappa}). \quad (3.2)$$

We must demonstrate that the assignment (3.2) is well-defined and has the properties needed of a net-isomorphism. To begin with, we list some useful auxiliary results.

Lemma 3.2

- (a) $\lim_{\lambda \rightarrow 0} \|(\omega^{(\lambda m)} - \omega^{(0)}) \upharpoonright \mathcal{R}^{(0)}(\mathcal{O})\| = 0$ for any double cone \mathcal{O} .
(b) Let $h \in \mathcal{D}(\mathbb{R}^{1+s})$ and $f \in \mathcal{D}(\mathbb{R}^s)$, and consider the function $\underline{W} : \mathbb{R}^+ \rightarrow \mathcal{A}^{(m)}$ given by

$$\underline{W}_\lambda := \int d^{1+s}x h(x) \alpha_{\lambda x}^{(m)} \circ \sigma_\lambda(W(f)), \quad \lambda > 0, \quad (3.3)$$

where the integral is to be understood in the weak sense. Then $\underline{W} \in \underline{\mathcal{A}}^{(m)}(\mathcal{O}_0)$ for some double cone \mathcal{O}_0 based on the time $t = 0$ plane. Moreover,

$$\lim_{\lambda \rightarrow 0} \sigma_\lambda^{-1}(\underline{W}_\lambda) = \int d^{1+s}x h(x) \alpha_x^{(0)}(W(f)) =: W_0 \quad (3.4)$$

in the strong-operator topology.

Proof of Lemma 3.2. (a) In view of the facts that $\omega^{(m)} \circ \sigma_\lambda = \omega^{(\lambda m)}$ and $\omega^{(0)}$ is invariant under the action of the dilations σ_λ , statement (a) is equivalent to

$$\lim_{\lambda \rightarrow 0} \|(\omega^{(m)} - \omega^{(0)}) \upharpoonright \mathcal{R}^{(0)}(\lambda \mathcal{O})\| = 0. \quad (3.5)$$

Since one has $\bigcap_{\lambda > 0} \mathcal{R}^{(0)}(\lambda \mathcal{O})^- = \mathbb{C}1$ on general grounds [13], relation (3.5) follows from an argument by Roberts [12] because $\omega^{(m)}$ is locally normal to $\omega^{(0)}$, cf. also [8].

(b) By construction, \underline{W} is obtained through convolution (with respect to the lifted action $\underline{\alpha}_x^{(m)}$) of the uniformly bounded function $\lambda \rightarrow \sigma_\lambda(W(f))$ and a test-function h . Thus it is strongly continuous with respect to the action of $\underline{\alpha}_x^{(m)}$. As a consequence of (2.13) and the support properties of f and h one observes that $\underline{W}_\lambda \in \mathcal{R}^{(m)}(\lambda \mathcal{O}_0)$ for some double cone \mathcal{O}_0 based on the time $t = 0$ plane. Hence, in view of the continuity properties of \underline{W} with respect to the translations there holds $\underline{W} \in \underline{\mathcal{A}}^{(m)}(\mathcal{O}_0)$.

For the final part of the statement we note that for all $x = (t, \mathbf{x}) \in \mathbb{R}^{1+s}$ one has

$$\begin{aligned} \|(\tau_x^{(m)} - \tau_x^{(0)})f\|_0^2 &= \|(\tau_t^{(m)} - \tau_t^{(0)})f\|_0^2 = \\ &= \left| \int \frac{d^s p}{2\mu_{0,\mathbf{p}}} (\cos(t\mu_{m,\mathbf{p}}) - \cos(t\mu_{0,\mathbf{p}})) \widetilde{\text{Re}} f(\mathbf{p}) - (\mu_{m,\mathbf{p}} \sin(t\mu_{m,\mathbf{p}}) - \mu_{0,\mathbf{p}} \sin(t\mu_{0,\mathbf{p}})) \widetilde{\text{Im}} f(\mathbf{p}) \right. \\ &\quad \left. + i\mu_{0,\mathbf{p}} (\cos(t\mu_{m,\mathbf{p}}) - \cos(t\mu_{0,\mathbf{p}})) \widetilde{\text{Im}} f(\mathbf{p}) + i\mu_{0,\mathbf{p}} \left(\frac{\sin(t\mu_{m,\mathbf{p}})}{\mu_{m,\mathbf{p}}} - \frac{\sin(t\mu_{0,\mathbf{p}})}{\mu_{0,\mathbf{p}}} \right) \widetilde{\text{Re}} f(\mathbf{p}) \right|^2. \end{aligned} \quad (3.6)$$

By an application of the dominated convergence theorem one concludes that one has $\lim_{\lambda \rightarrow 0} \|(\tau_t^{(\lambda m)} - \tau_t^{(0)})f\|_0^2 = 0$ uniformly in t on compact intervals. Employing a standard argument (e.g. [1, Prop. 5.2.4]), this implies that $W(\tau_x^{(\lambda m)} f)$ converges for $\lambda \rightarrow 0$, uniformly for x in any compact subset of \mathbb{R}^{1+s} , in the strong operator topology to $W(\tau_x^{(0)} f)$. The claimed statement follows from that. \square

Now we are in the position to show that the mapping ϕ given in (3.2) is well-defined. Let \mathcal{O} be any double cone. Take $\underline{A} \in \underline{\mathcal{A}}^{(m)}(\mathcal{O})$ and an arbitrary \underline{W} as in (3.3). Then

consider

$$\begin{aligned}
\omega_{0,\iota}^{(m)}(\underline{W}\underline{A}) &= \lim_{\kappa} \omega^{(m)}(\underline{W}_{\lambda_{\kappa}}\underline{A}_{\lambda_{\kappa}}) \\
&= \lim_{\kappa} \omega^{(\lambda_{\kappa}m)}(\sigma_{\lambda_{\kappa}}^{-1}(\underline{W}_{\lambda_{\kappa}})\sigma_{\lambda_{\kappa}}^{-1}(\underline{A}_{\lambda_{\kappa}})) \\
&= \lim_{\kappa} \omega^{(0)}(\sigma_{\lambda_{\kappa}}^{-1}(\underline{W}_{\lambda_{\kappa}})\sigma_{\lambda_{\kappa}}^{-1}(\underline{A}_{\lambda_{\kappa}})) \\
&= \lim_{\kappa} \omega^{(0)}(W_0\sigma_{\lambda_{\kappa}}^{-1}(\underline{A}_{\lambda_{\kappa}})).
\end{aligned} \tag{3.7}$$

Here, the second equality results from $\omega^{(m)} \circ \sigma_{\lambda} = \omega^{(\lambda m)}$, the third follows from Lemma 3.2(a) and the fact that $\sigma_{\lambda}^{-1}(\underline{W}_{\lambda})\sigma_{\lambda}^{-1}(\underline{A}_{\lambda}) \in \mathcal{R}^{(0)}(\mathcal{O}_0)$ for some fixed double cone \mathcal{O}_0 and all $\lambda > 0$, and the last equality is obtained from Lemma 3.2(b). Now notice that the linear combinations of the W_0 of the form of (3.4), as h and f vary, are dense in $\mathcal{R}^{(0)}(\mathcal{O})$. Since $\omega^{(0)}$ has the Reeh-Schlieder property (the corresponding GNS-vector is cyclic and separating for all local algebras $\mathcal{R}^{(0)}(\mathcal{O})$), it follows from (3.7) that $w - \lim_{\kappa} \omega_{\lambda_{\kappa}}^{-1}(\underline{A}_{\lambda_{\kappa}})$ exists. Moreover, since

$$\|\pi_{0,\iota}(\underline{A})\Omega_{0,\iota}\|^2 = \lim_{\kappa} \omega^{(0)}(\underline{A}_{\lambda_{\kappa}}^* \underline{A}_{\lambda_{\kappa}}) = \lim_{\kappa} \omega^{(0)}(\sigma_{\lambda_{\kappa}}^{-1}(\underline{A}_{\lambda_{\kappa}})^* \sigma_{\lambda_{\kappa}}^{-1}(\underline{A}_{\lambda_{\kappa}})), \tag{3.8}$$

the map ϕ in (3.2) maps 0 to 0 and hence is well-defined and linear. It is also *-preserving since the *-operation is continuous in the weak topology. One also has

$$\langle \Omega_{0,\iota}, \pi_{0,\iota}(\underline{A})\Omega_{0,\iota} \rangle = \omega^{(0)}(\phi(\pi_{0,\iota}(\underline{A}))), \quad \underline{A} \in \underline{\mathcal{A}}^{(m)}, \tag{3.9}$$

which already shows that ϕ connects $\omega_{0,\iota}^{(m)}$ and $\omega^{(0)}$.

In the next step, we will establish the intertwining relation

$$\phi(\alpha_x^{(m;0,\iota)}(\pi_{0,\iota}(\underline{A}))) = \alpha_x^{(0)}(\phi(\pi_{0,\iota}(\underline{A}))), \quad \underline{A} \in \underline{\mathcal{A}}^{(m)}. \tag{3.10}$$

Let us pick arbitrarily $\underline{A} \in \underline{\mathcal{A}}^{(m)}(\mathcal{O})$, and \underline{W} as in (3.3). Lemma 3.2(b) implies that $\lim_{\kappa} \alpha_x^{(\lambda_{\kappa}m)} \circ \sigma_{\lambda_{\kappa}}^{-1}(\underline{W}_{\lambda_{\kappa}}) = \alpha_x^{(0)}(W_0)$ in the strong operator topology. Making use of this we are led to the following chain of equations, valid for each $x \in \mathbb{R}^{1+s}$:

$$\begin{aligned}
\omega^{(0)}(W_0\sigma_{\lambda_{\kappa}}^{-1}(\alpha_{\lambda_{\kappa}x}^{(m)}(\underline{A}_{\lambda_{\kappa}}))) &= \omega^{(0)}(W_0\alpha_x^{(\lambda_{\kappa}m)}(\sigma_{\lambda_{\kappa}}^{-1}(\underline{A}_{\lambda_{\kappa}}))) \\
&= \omega^{(0)}(\sigma_{\lambda_{\kappa}}^{-1}(\underline{W}_{\lambda_{\kappa}})\alpha_x^{(\lambda_{\kappa}m)}(\sigma_{\lambda_{\kappa}}^{-1}(\underline{A}_{\lambda_{\kappa}}))) + O(1) \\
&= \omega^{(\lambda_{\kappa}m)}(\sigma_{\lambda_{\kappa}}^{-1}(\underline{W}_{\lambda_{\kappa}})\alpha_x^{(\lambda_{\kappa}m)}(\sigma_{\lambda_{\kappa}}^{-1}(\underline{A}_{\lambda_{\kappa}}))) + O(1) \\
&= \omega^{(\lambda_{\kappa}m)}(\alpha_{-x}^{(\lambda_{\kappa}m)} \circ \sigma_{\lambda_{\kappa}}^{-1}(\underline{W}_{\lambda_{\kappa}})\sigma_{\lambda_{\kappa}}^{-1}(\underline{A}_{\lambda_{\kappa}})) + O(1) \\
&= \omega^{(0)}(\alpha_{-x}^{(\lambda_{\kappa}m)} \circ \sigma_{\lambda_{\kappa}}^{-1}(\underline{W}_{\lambda_{\kappa}})\sigma_{\lambda_{\kappa}}^{-1}(\underline{A}_{\lambda_{\kappa}})) + O(1) \\
&= \omega^{(0)}(\alpha_{-x}^{(0)}(W_0)\sigma_{\lambda_{\kappa}}^{-1}(\underline{A}_{\lambda_{\kappa}})) + O(1) \\
&= \omega^{(0)}(W_0\alpha_x^{(0)} \circ \sigma_{\lambda_{\kappa}}^{-1}(\underline{A}_{\lambda_{\kappa}})) + O(1),
\end{aligned} \tag{3.11}$$

where in each line, $O(1)$ denotes some function of λ_{κ} which tends to 0 as $\lambda_{\kappa} \rightarrow 0$. For the first equality in (3.11) we have used eq. (2.8), the second follows from Lemma 3.2(b), the third from Lemma 3.2(a). One passes to the fourth equality by invariance

of the states under the corresponding action of the dynamics, and to the fifth by using again Lemma 3.2(a). The sixth equation is obtained from Lemma 3.2(b), and the last equality makes again use of the invariance of the state under the action of the translations. The desired relation (3.10) then results in the limit $\lambda_\kappa \rightarrow 0$, using again the fact that the span of the W_0 is dense in $\mathcal{R}^{(0)}(\mathcal{O})$, and that $\Omega^{(0)}$ has the Reeh-Schlieder property.

In a similar way one can establish the corresponding intertwining relations for the Lorentz transformations.

Now we want to show that the map ϕ is also multiplicative. At this point we make essential use of the fact that the massless free scalar field theory in $1+s$ -dimensional Minkowski spacetime satisfies, for $s = 2, 3$, the Haag-Swieca compactness condition [6]. We aim at proving

Lemma 3.3 *For all local $\underline{A}, \underline{B} \in \underline{\mathcal{A}}^{(m)}$ it holds that*

$$\phi(\pi_{0,t}(\underline{A})) \phi(\pi_{0,t}(\underline{B})) = \phi(\pi_{0,t}(\underline{A})\pi_{0,t}(\underline{B})). \quad (3.12)$$

Proof of Lemma 3.3. Suppose first that $\underline{A}, \underline{B}$ are localized in two separated double cones $\mathcal{O}_A, \mathcal{O}_B$ based on the time $t = 0$ plane. Then we will show that

$$\omega^{(0)}(\phi(\pi_{0,t}(\underline{A})) \phi(\pi_{0,t}(\underline{B}))) = \omega^{(0)}(\phi(\pi_{0,t}(\underline{A})\pi_{0,t}(\underline{B}))). \quad (3.13)$$

To this end we make use of the following result in [4] which is a consequence of the positivity of energy: For any given $\delta > 0$, there is some continuous, rapidly decaying function $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$\langle \Omega^{(0)}, AB\Omega^{(0)} \rangle = \langle \Omega^{(0)}, Af(H_0)B\Omega^{(0)} \rangle + \langle \Omega^{(0)}, Bf(H_0)A\Omega^{(0)} \rangle \quad (3.14)$$

holds for all pairs of operators A, B which satisfy $[\alpha_t^{(0)}(A), B] = 0$ for $-\delta < t < \delta$. Here, $\Omega^{(0)}$ denotes the GNS-vector of $\omega^{(0)}$ and H_0 the generator of the zero-mass dynamics (time-translations). Now $A_\kappa := \sigma_{\lambda_\kappa}^{-1}(\underline{A}_{\lambda_\kappa}) \in \mathcal{R}^{(0)}(\mathcal{O}_A)$ and $B_\kappa := \sigma_{\lambda_\kappa}^{-1}(B_\kappa) \in \mathcal{R}^{(0)}(\mathcal{O}_B)$ and consequently, because of locality, there is some $\delta > 0$ such that $[\alpha_t^{(0)}(A_\kappa), B_{\kappa'}] = 0$ for $|t| < \delta$ and all κ, κ' . Thus the aforementioned result applies to the effect that

$$\begin{aligned} \omega^{(0)}(\phi(\pi_{0,t}(\underline{A}))\phi(\pi_{0,t}(\underline{B}))) &= \lim_{\kappa} \lim_{\kappa'} \omega^{(0)}(A_\kappa B_{\kappa'}) \\ &= \lim_{\kappa} \lim_{\kappa'} \left(\langle \Omega^{(0)}, A_\kappa f(H_0)B_{\kappa'}\Omega^{(0)} \rangle + \langle \Omega^{(0)}, B_{\kappa'} f(H_0)A_\kappa\Omega^{(0)} \rangle \right) \end{aligned} \quad (3.15)$$

for some f as above. The crucial point here is that, since $w - \lim_{\kappa'} B_{\kappa'}$ exists and since $B_{\kappa'} \in \mathcal{R}^{(0)}(\mathcal{O}_B)$, there exist also the strong limits $s - \lim_{\kappa'} f(H_0)B_{\kappa'}\Omega^{(0)}$ and $s - \lim_{\kappa'} f(H_0)^* B_{\kappa'}^* \Omega^{(0)}$ because of the Haag-Swieca compactness for the massless free scalar field. This entails that one may pass from the double limit in (3.15) to the associated diagonal limit, i.e.

$$\lim_{\kappa} \left(\langle \Omega^{(0)}, A_\kappa f(H_0)B_\kappa\Omega^{(0)} \rangle + \langle \Omega^{(0)}, B_\kappa f(H_0)A_\kappa\Omega^{(0)} \rangle \right) \quad (3.16)$$

exists and equals the expression on the right hand side of (3.15). Since

$$\langle \Omega^{(0)}, A_\kappa f(H_0) B_\kappa \Omega^{(0)} \rangle + \langle \Omega^{(0)}, B_\kappa f(H_0) A_\kappa \Omega^{(0)} \rangle = \omega^{(0)}(A_\kappa B_\kappa) = \omega^{(0)}(\sigma_{\lambda_\kappa}(\underline{A}_{\lambda_\kappa} \underline{B}_{\lambda_\kappa})), \quad (3.17)$$

relation (3.13) is thus established.

In a second step, we shall extend the relation (3.13) to arbitrary local operators $\underline{A}, \underline{B}$. Here we make use of the general fact that any scaling limit theory is translation covariant and fulfills the spectrum condition [8]. For the case at hand this implies that the function

$$\mathbb{R}^{1+s} \ni x \mapsto G(x) := \omega_{0,\iota}(\underline{A} \underline{\alpha}_x^{(m)}(\underline{B})) \quad (3.18)$$

is the boundary value of an analytic function in the forward tube $\mathbb{R}^{1+s} + iV_+$. The same holds for the function

$$\mathbb{R}^{1+s} \ni x \mapsto F(x) := \langle \Omega^{(0)}, \phi(\pi_{0,\iota}(\underline{A})) \alpha_x^{(0)}(\phi(\pi_{0,\iota}(\underline{B}))) \Omega^{(0)} \rangle. \quad (3.19)$$

Since the operators \underline{A} and $\underline{\alpha}_x^{(m)}(\underline{B})$ are, for sufficiently large spacelike x , localized in disjoint double cones \mathcal{O}_A and \mathcal{O}_B as in the preceding step, we may then appeal to (3.13) to conclude that

$$\begin{aligned} F(x) &= \langle \Omega^{(0)}, \phi(\pi_{0,\iota}(\underline{A})) \phi(\pi_{0,\iota}(\underline{\alpha}_x^{(m)}(\underline{B}))) \Omega^{(0)} \rangle \\ &= \langle \Omega^{(0)}, \phi(\pi_{0,\iota}(\underline{A})) \pi_{0,\iota}(\underline{\alpha}_x^{(m)}(\underline{B})) \Omega^{(0)} \rangle \\ &= \langle \Omega_{0,\iota}, \pi_{0,\iota}(\underline{A} \underline{\alpha}_x^{(m)}(\underline{B})) \Omega_{0,\iota} \rangle = G(x) \end{aligned} \quad (3.20)$$

for some open set of translations x . By analyticity, we then obtain $F(x) = G(x)$ for all $x \in \mathbb{R}^s$, implying (3.13) for arbitrary local operators $\underline{A}, \underline{B}$.

Finally, (3.13) must be generalized to the operator identity (3.12). Let \underline{W} be of the form (3.3). Due to the strong-operator convergence of the families $\sigma_\lambda^{-1}(\underline{W}_\lambda)$ as $\lambda \rightarrow 0$, there holds for each $\underline{A} \in \underline{\mathcal{A}}^{(m)}(\mathcal{O})$ the following restricted form of multiplicativity of the map ϕ :

$$\phi(\pi_{0,\iota}(\underline{W})) \phi(\pi_{0,\iota}(\underline{A})) = \phi(\pi_{0,\iota}(\underline{W}) \pi_{0,\iota}(\underline{A})) \quad (3.21)$$

and similarly

$$\phi(\pi_{0,\iota}(\underline{A})) \phi(\pi_{0,\iota}(\underline{W})) = \phi(\pi_{0,\iota}(\underline{A}) \pi_{0,\iota}(\underline{W})). \quad (3.22)$$

Therefore, when \underline{W}' is also of the form (3.3) and when the \underline{A} and \underline{B} in (3.13) are replaced by $\underline{W} \cdot \underline{A}$ and $\underline{B} \cdot \underline{W}'$, respectively, we conclude that with W_0, W'_0 as in (3.4)

$$\begin{aligned} \omega^{(0)}(W_0 \phi(\pi_{0,\iota}(\underline{A})) \phi(\pi_{0,\iota}(\underline{B})) W'_0) &= \omega^{(0)}(\phi(\pi_{0,\iota}(\underline{W} \cdot \underline{A})) \phi(\pi_{0,\iota}(\underline{B} \cdot \underline{W}')))) \\ &= \omega^{(0)}(\phi(\pi_{0,\iota}(\underline{W} \cdot \underline{A})) \pi_{0,\iota}(\underline{B} \cdot \underline{W}')) = \omega^{(0)}(W_0 \phi(\pi_{0,\iota}(\underline{A}) \pi_{0,\iota}(\underline{B})) W'_0). \end{aligned} \quad (3.23)$$

Then the equation (3.12) results again from the fact that the linear span of the elements W_0, W'_0 is dense in the local von Neumann algebras $\mathcal{R}^{(0)}(\mathcal{O})$, and from the

Reeh-Schlieder property of $\omega^{(0)}$. This completes the proof of the multiplicativity of the map ϕ . \square

It remains to be shown that ϕ maps $\mathcal{A}_{0,\iota}^{(m)}(\mathcal{O})$ onto $\mathcal{A}^{(0)}(\mathcal{O})$ for all double cones \mathcal{O} . We will prove this first for all \mathcal{O}_0 based on the time $t = 0$ plane.

Let $A \in \mathcal{A}^{(0)}(\mathcal{O}_0)$, choose some function $h \in \mathcal{D}(\mathbb{R}^{1+s})$, and define the function $\mathbb{R}^+ \ni \lambda \mapsto \underline{A}_\lambda^{(h)}$ by

$$\underline{A}_\lambda^{(h)} := \int d^{1+s}x h(x) \alpha_{\lambda x}^{(m)}(\sigma_\lambda(A)). \quad (3.24)$$

Then $\underline{A}^{(h)} \in \underline{\mathcal{A}}^{(m)}(\mathcal{O}_h)$ where \mathcal{O}_h shrinks to \mathcal{O}_0 when the support of h is shrunk to $\{0\}$. We observe that

$$\sigma_\lambda^{-1}(\underline{A}_\lambda^{(h)}) = \int d^{1+s}x h(x) \alpha_x^{(\lambda m)}(A), \quad (3.25)$$

from which we now obtain the following chain of equations, where $\underline{W} \in \underline{\mathcal{A}}^{(m)}$ is of the form (3.3), and W_0 relates to it as in (3.4).

$$\begin{aligned} \lim_{\lambda \rightarrow 0} \omega^{(0)}(W_0 \sigma_\lambda^{-1}(\underline{A}_\lambda^{(h)})) &= \lim_{\lambda \rightarrow 0} \omega^{(0)}(\sigma_\lambda^{-1}(\underline{W}_\lambda) \sigma_\lambda^{-1}(\underline{A}_\lambda^{(h)})) \\ &= \lim_{\lambda \rightarrow 0} \omega^{(\lambda m)}(\sigma_\lambda^{-1}(\underline{W}_\lambda) \sigma_\lambda^{-1}(\underline{A}_\lambda^{(h)})) \\ &= \lim_{\lambda \rightarrow 0} \int d^{1+s}x h(x) \omega^{(\lambda m)}(\alpha_{-x}^{(\lambda m)} \circ \sigma_\lambda^{-1}(\underline{W}_\lambda) \cdot A) \\ &= \lim_{\lambda \rightarrow 0} \int d^{1+s}x h(x) \omega^{(0)}(\alpha_{-x}^{(\lambda m)} \circ \sigma_\lambda^{-1}(\underline{W}_\lambda) \cdot A) \\ &= \int d^{1+s}x h(x) \omega^{(0)}(\alpha_{-x}^{(0)}(W_0) A). \end{aligned} \quad (3.26)$$

In the preceding chain, the first equality is derived by means of Lemma 3.2(b), the second by Lemma 3.2(a), and the third uses the invariance of $\omega^{(\lambda m)}$ under the translations $\alpha_x^{(\lambda m)}$. The fourth equation is implied by Lemma 3.2(a) and the last one follows on account of Lemma 3.2(b). Falling back on the by now familiar argument that the span of elements W_0 is weakly dense in $\mathcal{R}^{(0)}(\mathcal{O}_0)$, the just obtained equalities entail that

$$\phi(\pi_{0,\iota}(\underline{A}^{(h)})) = w - \lim_{\lambda \rightarrow 0} \sigma_\lambda^{-1}(\underline{A}_\lambda^{(h)}) = \int d^{1+s}x h(x) \alpha_x^{(0)}(A). \quad (3.27)$$

Since A and h were arbitrary we see that $\phi(\mathcal{A}_{0,\iota}^{(m)})\Omega$ spans $\mathcal{H}^{(0)}$, the standard Hilbert space. Hence, because of the algebraic properties of ϕ and (3.9), ϕ is in fact given by the adjoint action of a unitary $\mathcal{H}_{0,\iota} \rightarrow \mathcal{H}^{(0)}$ defined through the assignment $\pi_{0,\iota}(\underline{A})\Omega_{0,\iota} \mapsto \phi(\pi_{0,\iota}(\underline{A}))\Omega$. Therefore, ϕ is in particular injective.

If one takes now a sequence h_n of test-functions approaching the Dirac-measure δ , one has $\phi(\pi_{0,\iota}(\underline{A}^{(h_n)})) \rightarrow A$ for $n \rightarrow \infty$ in norm since $A \in \mathcal{A}^{(0)}(\mathcal{O}_0)$ is, by the very definition of the algebra $\mathcal{A}^{(0)}(\mathcal{O}_0)$, an element on which the translations $\alpha_x^{(0)}$, $x \in \mathbb{R}^{1+s}$, act strongly continuously. This shows that, for each pair $\mathcal{O}_0, \mathcal{O}_1$ of double cones based on the time $t = 0$ hyperplane with $\overline{\mathcal{O}_0} \subset \mathcal{O}_1$, we have $\mathcal{A}^{(0)}(\mathcal{O}_0) \subset \phi(\mathcal{A}_{0,\iota}^{(m)}(\mathcal{O}_1))$. On

the other hand, for each \mathcal{O}_1 there holds the inclusion $\phi(\mathcal{A}_{0,\iota}^{(m)}(\mathcal{O}_1)) \subset \mathcal{A}^{(0)}(\mathcal{O}_1)$ in view of the intertwining relation (3.10) and the fact that $\phi(\mathcal{A}_{0,\iota}^{(m)}(\mathcal{O}_1)) \subset \mathcal{R}^{(0)}(\mathcal{O}_1)$. Hence

$$\mathcal{A}^{(0)}(\mathcal{O}_0) \subset \phi(\mathcal{A}_{0,\iota}^{(m)}(\mathcal{O}_1)) \subset \mathcal{A}^{(0)}(\mathcal{O}_1), \quad (3.28)$$

and because of the injectivity of ϕ we conclude that

$$\mathcal{A}^{(0)}(\mathcal{O}_0) \subset \phi \left(\bigcap_{\mathcal{O}_1 \supset \overline{\mathcal{O}_0}} \mathcal{A}_{0,\iota}^{(m)}(\mathcal{O}_1) \right) \subset \bigcap_{\mathcal{O}_1 \supset \overline{\mathcal{O}_0}} \mathcal{A}^{(0)}(\mathcal{O}_1). \quad (3.29)$$

The required equality $\phi(\mathcal{A}_{0,\iota}^{(m)}(\mathcal{O}_0)) = \mathcal{A}^{(0)}(\mathcal{O}_0)$ follows now from the outer continuity of the nets $\mathcal{O} \rightarrow \mathcal{A}^{(0)}(\mathcal{O})$ and $\mathcal{O} \rightarrow \mathcal{A}_{0,\iota}^{(m)}(\mathcal{O})$. Due to the intertwining property of ϕ for the Poincaré transformations, cf. (3.10), we immediately obtain

$$\begin{aligned} \phi(\mathcal{A}_{0,\iota}^{(m)}(\Lambda\mathcal{O}_0 + x)) &= \phi(\alpha_{\Lambda,x}^{(m;0,\iota)}(\mathcal{A}_{0,\iota}^{(m)}(\mathcal{O}_0))) \\ &= \alpha_{\Lambda,x}^{(0)}(\phi(\mathcal{A}_{0,\iota}^{(m)}(\mathcal{O}_0))) = \mathcal{A}^{(0)}(\Lambda\mathcal{O}_0 + x) \end{aligned} \quad (3.30)$$

for all \mathcal{O}_0 based on the time $t = 0$ plane and all Λ, x . Consequently the equality $\phi(\mathcal{A}_{0,\iota}^{(m)}(\mathcal{O})) = \mathcal{A}^{(0)}(\mathcal{O})$ holds also for all double cones \mathcal{O} .

This completes the proof of Theorem 3.1.

4 Construction of charged states for $s = 1$

As we shall see, the properties of the scaling limit theory of the free massive scalar field in two-dimensional Minkowski-spacetime are in several respects different from the case in three- and four-dimensional Minkowski-spacetime. First, while the vacuum states appearing in the scaling limit are always pure in $s \geq 2$ dimensions [8], the present model in $s = 1$ dimension provides an example where these states are mixed. Second, there appear charged states in the scaling limit whose restrictions to the algebras of both, the right and left component of the spacelike complement of a double cone region, coincide with the vacuum. But their restriction to the algebra of the full spacelike complement is disjoint from the vacuum. Hence these states carry a gauge charge in the sense of [5]. As was discussed in [3], there holds Gauss' law for this charge. These results will be formulated more precisely in the following theorem.

Theorem 4.1 *Let $\omega_{0,\iota}^{(m)}$ be any scaling limit state of the theory of the free scalar field of mass $m > 0$ in two-dimensional Minkowski-spacetime. Then for the corresponding scaling limit theory $(\mathcal{A}_{0,\iota}^{(m)}, \alpha^{(m;0,\iota)}, \omega_{0,\iota}^{(m)})$ it holds that:*

- (a) $\mathcal{A}_{0,\iota}^{(m)}$ possesses a non-trivial center.
- (b) There are charged states $\omega_{q,\iota}^{(m)}$ on $\mathcal{A}_{0,\iota}^{(m)}$ which are locally normal to $\omega_{0,\iota}^{(m)}$ and have moreover the following properties:¹

¹As usual, the C^* -algebras corresponding to unbounded regions are generated by the algebras associated with all double cones which are contained in the respective region.

(b.i) $\omega_{q,\iota}^{(m)} \upharpoonright \mathcal{A}_{0,\iota}^{(m)}(\mathcal{O}^{(\pm)}) = \omega_{0,\iota}^{(m)} \upharpoonright \mathcal{A}_{0,\iota}^{(m)}(\mathcal{O}^{(\pm)})$ for sufficiently large double cones \mathcal{O} , where $\mathcal{O}^{(\pm)}$ denotes the right/left component of \mathcal{O}' .

(b.ii) $\omega_{q,\iota}^{(m)} \upharpoonright \mathcal{A}_{0,\iota}^{(m)}(\mathcal{O}')$ is disjoint from $\omega_{0,\iota}^{(m)} \upharpoonright \mathcal{A}_{0,\iota}^{(m)}(\mathcal{O}')$ for all double cones \mathcal{O} .

(b.iii) In the GNS-representation induced by $\omega_{q,\iota}^{(m)}$ the translations $\alpha_{\mathbb{R}^2}^{(m;0,\iota)}$ are implemented by a continuous unitary representation satisfying the spectrum condition.

The remainder of this section is devoted to the proof of these statements.

Let $m > 0$ be given. We choose the Fock-representation induced by $\omega^{(m)}$ as our standard representation. The corresponding GNS-vacuum vector will be denoted by $\Omega^{(m)}$.

To prove (a), we define for arbitrary $h \in \mathcal{D}(\mathbb{R}^2)$ and real-valued $f \in \mathcal{D}(\mathbb{R})$,

$$\underline{C}_\lambda^{(h)} := \int d^2x h(x) \alpha_{\lambda x}^{(m)} \left(W(|\ln \lambda|^{-1/2} \delta_\lambda f) \right), \quad \lambda > 0. \quad (4.1)$$

Then $\lambda \mapsto \underline{C}_\lambda^{(h)}$ is contained in $\mathcal{A}^{(m)}(\mathcal{O})$ for a suitable double cone \mathcal{O} . Moreover, there hold the following relations:

$$\lim_{\lambda \rightarrow 0} \langle \Omega^{(m)}, \underline{C}_\lambda^{(h)} \Omega^{(m)} \rangle = \int d^2x h(x) \cdot e^{-(1/2)|\tilde{f}(0)|^2}, \quad (4.2)$$

$$\lim_{\lambda \rightarrow 0} \langle \Omega^{(m)}, \underline{C}_\lambda^{(h)} \underline{C}_\lambda^{(h)} \Omega^{(m)} \rangle = \left(\int d^2x h(x) \right)^2 \cdot e^{-2|\tilde{f}(0)|^2}, \quad (4.3)$$

$$\lim_{\lambda \rightarrow 0} \langle \Omega^{(m)}, \underline{C}_\lambda^{(h)*} \underline{C}_\lambda^{(h)} \Omega^{(m)} \rangle = \left| \int d^2x h(x) \right|^2. \quad (4.4)$$

Equation (4.2) can be derived as follows. Inserting the expression for $\underline{C}_\lambda^{(h)}$ into the vacuum state yields

$$\langle \Omega^{(m)}, \underline{C}_\lambda^{(h)} \Omega^{(m)} \rangle = \int d^2x h(x) \cdot e^{-(1/2) \| |\ln \lambda|^{-1/2} \delta_\lambda f \|_m^2}, \quad (4.5)$$

where we used the invariance of the vacuum under translations. For the term in the exponential we obtain by partial integration

$$\begin{aligned} \| |\ln \lambda|^{-1/2} \delta_\lambda f \|_m^2 &= |\ln \lambda|^{-1} \cdot \int_0^\infty \frac{d\mathbf{p}}{\sqrt{\mathbf{p}^2 + (\lambda m)^2}} |\tilde{f}(\mathbf{p})|^2 \\ &= -|\ln \lambda|^{-1} \cdot \ln(\lambda m) |\tilde{f}(0)|^2 - |\ln \lambda|^{-1} \cdot \int_0^\infty d\mathbf{p} \ln(\mathbf{p} + \sqrt{\mathbf{p}^2 + (\lambda m)^2}) \frac{d}{d\mathbf{p}} |\tilde{f}(\mathbf{p})|^2, \end{aligned} \quad (4.6)$$

where we made use of the fact that $\mathbf{p} \mapsto |\tilde{f}(\mathbf{p})|^2$ is symmetric since f is real. Since f is a test-function, relation (4.2) then follows. For the proof of (4.3) and (4.4) one has also to make use of the Weyl-relations and the specific form of the action of the translations on the Weyl-operators. Otherwise the reasoning is completely analogous.

Now let $(\pi_{0,\iota}, \mathcal{H}_{0,\iota}, \Omega_{0,\iota})$ denote the GNS-representation of $\omega_{0,\iota}^{(m)}$, and $C_{0,\iota}^{(h)} := \pi_{0,\iota}(\underline{C}^{(h)})$. We want to show that $C_{0,\iota}^{(h)}$ is contained in the center of $\mathcal{A}_{0,\iota}^{(m)}$ (for any choice of h and f). Due to locality it is sufficient to show that $C_{0,\iota}^{(h)}$ is translation

invariant, i.e. $\alpha_x^{(m;0,\iota)}(C_{0,\iota}^{(h)}) = C_{0,\iota}^{(h)}$ for all x . From the definition of $\underline{C}^{(h)}$ it follows that $\underline{\alpha}_x^{(m)}(\underline{C}^{(h)}) = \underline{C}^{(h_x)}$ where $h_x(y) = h(y - x)$ denotes the translate of h . Moreover, it is easily seen that $\underline{C}^{(h_x)} - \underline{C}^{(h)} = \underline{C}^{(h_x-h)}$. Hence we have, by (4.4),

$$\begin{aligned} ||(\alpha_x^{(m;0,\iota)}(C_{0,\iota}^{(h)}) - C_{0,\iota}^{(h)})\Omega_{0,\iota}||^2 &= ||C_{0,\iota}^{(h_x-h)}\Omega_{0,\iota}||^2 \\ &= \lim_{\lambda \rightarrow 0} \langle \Omega^{(m)}, \underline{C}_\lambda^{(h_x-h)*} \underline{C}_\lambda^{(h_x-h)} \Omega^{(m)} \rangle \\ &= \left| \int d^2y (h_x(y) - h(y)) \right|^2 = 0. \end{aligned} \quad (4.7)$$

Since $C_{0,\iota}^{(h_x-h)}$ is contained in $\mathcal{A}_{0,\iota}^{(m)}(\mathcal{O}_x)$ for some double cone \mathcal{O}_x , and since $\Omega_{0,\iota}$ is separating for these local algebras [8, Lemma 6.1], the last equality entails that $C_{0,\iota}^{(h_x-h)} = 0$. Thus $C_{0,\iota}^{(h)}$ is invariant under translations and hence lies in the center of $\mathcal{A}_{0,\iota}^{(m)}$. Finally, (4.2) and (4.3) imply that $C_{0,\iota}^{(h)}$ is different from a multiple of 1 in the case that $\int d^2x h(x) \neq 0$ and $\int d\mathbf{x} f(\mathbf{x}) \neq 0$, because then one has

$$\begin{aligned} \langle \Omega_{0,\iota}, C_{0,\iota}^{(h)} \Omega_{0,\iota} \rangle^2 &= \left(\int d^2x h(x) \right)^2 \cdot e^{-|\tilde{f}(0)|^2} \\ &\neq \left(\int d^2x h(x) \right)^2 \cdot e^{-2|\tilde{f}(0)|^2} = \langle \Omega_{0,\iota}, C_{0,\iota}^{(h)} C_{0,\iota}^{(h)} \Omega_{0,\iota} \rangle. \end{aligned} \quad (4.8)$$

Part (a) of the theorem is thereby proved.

We note that the appearance of a non-trivial center in the scaling limit may be attributed to the fact that the massive free field in two dimensions contains an anomalous classical part, cf. [3].

Let us now turn to the proof of part (b) of the theorem. For given $h \in \mathcal{D}(\mathbb{R}^2)$ and $f \in \mathcal{D}(\mathbb{R})$ we define $\underline{W}_\lambda(f) := \sigma_\lambda(W(f))$ and

$$\underline{W}_\lambda^{(h)}(f) := \int d^2x h(x) \alpha_{\lambda x}^{(m)}(\underline{W}_\lambda(f)). \quad (4.9)$$

Then $\lambda \mapsto \underline{W}_\lambda^{(h)}(f)$ is contained in $\mathcal{A}^{(m)}(\mathcal{O})$ for some double cone \mathcal{O} . Whereas the scaling algebra contains only these regularized Weyl-operators, one can recover from them the non-regularized Weyl-operators in the scaling limit. This is shown in the subsequent lemma.

Lemma 4.2 *Let $(\pi_{0,\iota}, \mathcal{H}_{0,\iota}, \Omega_{0,\iota})$ be the GNS-representation of $\omega_{0,\iota}^{(m)}$, and let h_n be any sequence of smooth functions on \mathbb{R}^2 with $h_n(x) \geq 0$, $h_n(x) = 0$ for $|x| \geq n^{-1}$ and $\int d^2x h_n(x) = 1$. Then for each $f \in \mathcal{D}(\mathbb{R})$,*

$$W_{0,\iota}(f) := \lim_{n \rightarrow \infty} \pi_{0,\iota}(\underline{W}^{(h_n)}(f)) \quad (4.10)$$

exists in the strong operator topology and, for the net λ_κ , $\kappa \in \mathbb{K}$, used in the construction of $\omega_{0,\iota}^{(m)}$,

$$\lim_{\kappa} \omega^{(m)}(\underline{W}_{\lambda_\kappa}(f) \underline{A}_{\lambda_\kappa} \underline{W}_{\lambda_\kappa}(g)) = \langle \Omega_{0,\iota}, W_{0,\iota}(f) \pi_{0,\iota}(\underline{A}) W_{0,\iota}(g) \Omega_{0,\iota} \rangle \quad (4.11)$$

for all $f, g \in \mathcal{D}(\mathbb{R})$, $\underline{A} \in \mathcal{A}^{(m)}$. Moreover, there hold the Weyl relations

$$W_{0,\iota}(f) W_{0,\iota}(g) = e^{-i\sigma(f,g)/2} W_{0,\iota}(f+g), \quad f, g \in \mathcal{D}(\mathbb{R}). \quad (4.12)$$

Proof of Lemma 4.2. First we show that for $f \in \mathcal{D}(\mathbb{R})$ one has

$$\sup_{1 > \lambda > 0} \|(\alpha_{\lambda x}^{(m)}(\underline{W}_\lambda(f)) - \underline{W}_\lambda(f))\Omega^{(m)}\| \rightarrow 0 \quad \text{for } x \rightarrow 0. \quad (4.13)$$

Since $\omega^{(m)} \circ \sigma_\lambda = \omega^{(\lambda m)}$ and in view of (2.8), this amounts to showing that

$$\sup_{1 > \lambda > 0} \|(\alpha_x^{(\lambda m)}(W(f)) - W(f))\Omega^{(\lambda m)}\| \rightarrow 0 \quad \text{for } x \rightarrow 0. \quad (4.14)$$

For the latter it is sufficient to show that

$$\sup_{1 > \lambda > 0} \|\tau_x^{(\lambda m)} f - f\|_{\lambda m} \rightarrow 0 \quad \text{for } x \rightarrow 0. \quad (4.15)$$

But for all $x = (t, \mathbf{x}) \in \mathbb{R}^2$ and $1 > \lambda > 0$ one obtains

$$\begin{aligned} & \|\tau_x^{(\lambda m)} f - f\|_{\lambda m}^2 \\ &= \int \frac{d\mathbf{p}}{2\mu_{\lambda m, \mathbf{p}}} \left| (e^{it\mu_{\lambda m, \mathbf{p}} - i\mathbf{p}\mathbf{x}} - 1) (\widetilde{\text{Re}}f(\mathbf{p}) + i\mu_{\lambda m, \mathbf{p}} \widetilde{\text{Im}}f(\mathbf{p})) \right|^2 \\ &\leq 2 \int d\mathbf{p} (|t| + |x|)^2 \mu_{m, \mathbf{p}} \left(|\widetilde{\text{Re}}f(\mathbf{p})| + \mu_{m, \mathbf{p}} |\widetilde{\text{Im}}f(\mathbf{p})| \right)^2 \leq \text{const} (|t| + |x|)^2, \end{aligned} \quad (4.16)$$

hence (4.13) is established.

Now let $f \in \mathcal{D}(\mathbb{R})$ be given. For $\varepsilon > 0$, we choose some $n \in \mathbb{N}$ so that

$$\sup_{1 > \lambda > 0} \|(\alpha_{\lambda x}^{(m)}(\underline{W}_\lambda(f)) - \underline{W}_\lambda(f))\Omega^{(m)}\| < \varepsilon/4 \quad (4.17)$$

for $|x| < n^{-1}$. Then for all $j, k > n$ we have

$$\begin{aligned} & \|(\pi_{0, \iota}(\underline{W}_\lambda^{(h_j)}(f)) - \underline{W}_\lambda^{(h_k)}(f))\Omega_{0, \iota}\| \\ &= \lim_{\kappa} \|(\underline{W}_{\lambda_\kappa}^{(h_j)}(f) - \underline{W}_{\lambda_\kappa}^{(h_k)}(f))\Omega^{(m)}\| \leq \sup_{1 > \lambda > 0} \|(\underline{W}_\lambda^{(h_j)}(f) - \underline{W}_\lambda^{(h_k)}(f))\Omega^{(m)}\| \\ &\leq 2 \int d^2x (h_j(x) + h_k(x)) \sup_{1 > \lambda > 0} \|(\alpha_{\lambda x}^{(m)}(\underline{W}_\lambda(f)) - \underline{W}_\lambda(f))\Omega^{(m)}\| < \varepsilon. \end{aligned} \quad (4.18)$$

Here we used that the integral equals 2 and the support of h_j is contained in the ball of radius n^{-1} for $j > n$. Since $\Omega_{0, \iota}$ is separating for the local von Neumann algebras $\mathcal{A}_{0, \iota}^{(m)}(\mathcal{O})^-$, this last estimate shows that $\pi_{0, \iota}(\underline{W}_\lambda^{(h_j)}(f))$ is strongly convergent to an element in $\mathcal{A}_{0, \iota}^{(m)}(\mathcal{O})^-$ for some double cone \mathcal{O} as $j \rightarrow \infty$.

To prove (4.11), we note that (4.17) also implies that $\sup_{1 > \lambda > 0} \|(\underline{W}_\lambda^{(h_j)}(f) - \underline{W}_\lambda(f))\Omega\| < \varepsilon/2$, for all $j > n$. Hence one obtains that

$$\sup_{1 > \lambda > 0} |\omega^{(m)}(\underline{W}_\lambda^{(h_j)}(f) \underline{A}_\lambda \underline{W}_\lambda^{(h_j)}(g)) - \omega^{(m)}(\underline{W}_\lambda(f) \underline{A}_\lambda \underline{W}_\lambda(g))| \quad (4.19)$$

can be made arbitrarily small for sufficiently large j . Relation (4.11) is implied by this fact.

Now let us turn to proving (4.12). Let $f, g \in \mathcal{D}(\mathbb{R})$, then $W_{0,\iota}(f)$ and $W_{0,\iota}(g)$ are contained in $\mathcal{A}_{0,\iota}^{(m)}(\mathcal{O})^-$ for some double cone \mathcal{O} . With the help of (4.11), when $\mathcal{O}_1 \in \mathcal{O}'$, we obtain for all $\underline{A} \in \underline{\mathcal{A}}^{(m)}(\mathcal{O}_1)$,

$$\begin{aligned} \langle \Omega_{0,\iota}, \pi_{0,\iota}(\underline{A}) W_{0,\iota}(f) W_{0,\iota}(g) \Omega_{0,\iota} \rangle &= \langle \Omega_{0,\iota}, W_{0,\iota}(f) \pi_{0,\iota}(\underline{A}) W_{0,\iota}(g) \Omega_{0,\iota} \rangle \\ &= \lim_{\kappa} \omega^{(m)}(\underline{W}_{\lambda_\kappa}(f) \underline{A}_{\lambda_\kappa} \underline{W}_{\lambda_\kappa}(g)) \\ &= \lim_{\kappa} \omega^{(m)}(\underline{A}_{\lambda_\kappa} \underline{W}_{\lambda_\kappa}(f) \underline{W}_{\lambda_\kappa}(g)) \\ &= \lim_{\kappa} \omega^{(m)}(\underline{A}_{\lambda_\kappa} e^{-i\sigma(f,g)/2} \underline{W}_{\lambda_\kappa}(f+g)) \\ &= \langle \Omega_{0,\iota}, \pi_{0,\iota}(\underline{A}) e^{-i\sigma(f,g)/2} W_{0,\iota}(f+g) \Omega_{0,\iota} \rangle, \end{aligned} \quad (4.20)$$

where we made use of locality and the Weyl-relations. Since $\Omega_{0,\iota}$ is cyclic for the C^* -algebra generated by $\{\mathcal{A}_{0,\iota}^{(m)}(\mathcal{O}_1), \mathcal{O}_1 \subset \mathcal{O}'\}$ [8, Lemma 6.1] and separating for $\mathcal{A}_{0,\iota}^{(m)}(\mathcal{O})^-$ this establishes (4.12). \square

On the basis of the last lemma, we can define the following two families of unitary operators on the scaling-limit Hilbert space $\mathcal{H}_{0,\iota}$. First, we pick some sequence of real-valued $f_n \in \mathcal{D}(\mathbb{R})$ where $f_n(\mathbf{x}) = q$, $|\mathbf{x}| < n$, for an arbitrary real number q and define

$$Y_q^{(n)} := W_{0,\iota}(if_n), \quad n \in \mathbb{N}. \quad (4.21)$$

Secondly, for any choice of a real-valued $w \in \mathcal{D}(\mathbb{R})$ with $w(\mathbf{x}) = 1$ for $|\mathbf{x}| \leq 1$, we consider the sequence of test-functions $g_n(\mathbf{x}) := \partial_{\mathbf{x}} w(\mathbf{x}/n)$, $n \in \mathbb{N}$, and define

$$Z_w^{(n)}(r) := W_{0,\iota}(r g_n), \quad r \in \mathbb{R}. \quad (4.22)$$

In the following lemma we collect some properties of these unitaries.

Lemma 4.3 *It holds that*

$$\lim_{n \rightarrow \infty} \langle Y_q^{(n)} \Omega_{0,\iota}, \pi_{0,\iota}(\underline{A}) Y_q^{(n)} \Omega_{0,\iota} \rangle = \langle \Omega_{0,\iota}, \pi_{0,\iota}(\underline{A}) \Omega_{0,\iota} \rangle, \quad \underline{A} \in \underline{\mathcal{A}}^{(m)}, \quad (4.23)$$

and

$$\langle \Omega_{0,\iota}, Z_w^{(n)}(r) \Omega_{0,\iota} \rangle = e^{-(r^2/4) \cdot \int d\mathbf{k} |\mathbf{k}| |\tilde{w}(\mathbf{k})|^2}, \quad r \in \mathbb{R}, \quad n \in \mathbb{N}. \quad (4.24)$$

Proof of Lemma 4.3. Let $a > 0$ and $\underline{A} \in \underline{\mathcal{A}}^{(m)}(\mathcal{O}_a)$, where \mathcal{O}_a is the double cone based on the interval $(-a, a)$ in the time $t = 0$ plane. For $n > a$ we obtain by (4.11)

$$\begin{aligned} \langle Y_q^{(n)} \Omega_{0,\iota}, \pi_{0,\iota}(\underline{A}) Y_q^{(n)} \Omega_{0,\iota} \rangle &= \lim_{\kappa} \omega^{(m)}(\underline{W}_{\lambda_\kappa}(if_n)^* \underline{A}_{\lambda_\kappa} \underline{W}_{\lambda_\kappa}(if_n)) \\ &= \lim_{\kappa} \omega^{(m)}(W(if_n)^* \underline{A}_{\lambda_\kappa} W(if_n)) \\ &= \langle \Omega_{0,\iota}, \pi_{0,\iota}(\underline{A}) \Omega_{0,\iota} \rangle, \end{aligned} \quad (4.25)$$

where in the second equality we have used that

$$\begin{aligned} \underline{W}_\lambda(if_n)^* \underline{A}_\lambda \underline{W}_\lambda(if_n) &= W(\delta_\lambda if_n)^* \underline{A}_\lambda W(\delta_\lambda if_n) \\ &= W(if_n)^* \underline{A}_\lambda W(if_n) \end{aligned} \quad (4.26)$$

for $1 > \lambda > 0$. This holds because both if_n and $\delta_\lambda if_n$ are equal to iq in a neighborhood of the closure of $(-\lambda a, \lambda a)$, and so we see that $if_n = i\chi_n^{(\lambda)} + \delta_\lambda if_n$ with a real-valued function $\chi_n^{(\lambda)} \in \mathcal{D}(\mathbb{R})$ having support outside of $(-\lambda a, \lambda a)$. As $\underline{A}_\lambda \in \underline{\mathcal{A}}^{(m)}(\lambda \mathcal{O}_a)$, it commutes with $W(i\chi_n^{(\lambda)})$ and thus the last identity of (4.26) results from the Weyl-relations. This argument may also be used to show that $W(if_n)^* \underline{A}_\lambda W(if_n)$, $1 > \lambda > 0$, is independent of $n > a$. The last equality of (4.25) is then obtained for fixed $n > a$ since the scaling limit of a state locally normal to $\omega^{(m)}$ is equal to the scaling limit of $\omega^{(m)}$, cf. [8, Cor. 4.2]. This proves (4.23).

Equation (4.24) is obtained with the help of (4.11) by a straightforward computation,

$$\begin{aligned} \langle \Omega_{0,\iota}, Z_w^{(n)}(r) \Omega_{0,\iota} \rangle &= \lim_{\kappa} \omega^{(m)}(\underline{W}_{\lambda\kappa}(g_n)) \\ &= \lim_{\kappa} e^{-\frac{1}{2} \|g_n\|_{\lambda\kappa}^2} \\ &= e^{-(r^2/4) \cdot \int d\mathbf{k} |\mathbf{k}| |\tilde{w}(\mathbf{k})|^2}. \end{aligned} \quad (4.27) \quad \square$$

We note that by construction $g_n(\mathbf{x}) = 0$ for $|\mathbf{x}| < n$, hence each weak limit point of $Z^{(n)}(r)$, as $n \rightarrow \infty$, lies in the center of $\mathcal{A}_{0,\iota}^{(m)-}$. Drawing on (4.24) it is straightforward to show that these central elements are also different from multiples of the identity.

We construct now a state $\omega_{q,\iota}^{(m)}$ with the properties as claimed in part (b) of the Theorem. For this purpose, we consider the operators

$$V_q^{(n)} := W_{0,\iota}(iu_n), \quad n \in \mathbb{N}, \quad (4.28)$$

where the sequence of real-valued $u_n \in \mathcal{D}(\mathbb{R})$ is required to have, for some fixed $a > 0$, the following properties:

$$u_n(\mathbf{x}) = \begin{cases} 0, & \mathbf{x} \leq -a, \\ \text{independent of } n, & |\mathbf{x}| \leq a, \\ q, & a \leq \mathbf{x} \leq na. \end{cases} \quad (4.29)$$

Here $q \neq 0$ is some real number (“charge”) which will be kept fixed in the following. With the help of the operators $V_q^{(n)}$ we define the state

$$\omega_{q,\iota}^{(m)}(\pi_{0,\iota}(\underline{A})) := \lim_n \langle V_q^{(n)} \Omega_{0,\iota}, \pi_{0,\iota}(\underline{A}) V_q^{(n)} \Omega_{0,\iota} \rangle, \quad \underline{A} \in \underline{\mathcal{A}}^{(m)}. \quad (4.30)$$

An argument similar to the one used to prove (4.23) shows that for each double cone \mathcal{O} in \mathbb{R}^2 there exists some number n_0 such that, for each $\underline{A} \in \underline{\mathcal{A}}^{(m)}(\mathcal{O})$, the expression $\langle V_q^{(n)} \Omega_{0,\iota}, \pi_{0,\iota}(\underline{A}) V_q^{(n)} \Omega_{0,\iota} \rangle$ is independent of n for $n > n_0$. Therefore, the state $\omega_{q,\iota}^{(m)}$ exists and is locally normal to the scaling limit vacuum $\omega_{0,\iota}^{(m)}$.

If \mathcal{O} is a double cone which is contained in $\mathcal{O}_a^{(+)}$, we obtain for all $\underline{A} \in \underline{\mathcal{A}}^{(m)}(\mathcal{O})$ and sufficiently large n ,

$$\langle V_q^{(n)} \Omega_{0,\iota}, \pi_{0,\iota}(\underline{A}) V_q^{(n)} \Omega_{0,\iota} \rangle = \langle Y_q^{(n)} \Omega_{0,\iota}, \pi_{0,\iota}(\underline{A}) Y_q^{(n)} \Omega_{0,\iota} \rangle. \quad (4.31)$$

Again, this can easily be checked by an argument similar to that establishing (4.23). In view of Lemma 4.3 we conclude that

$$\omega_{q,\iota}^{(m)} \upharpoonright \mathcal{A}_{0,\iota}^{(m)}(\mathcal{O}_a^{(+)}) = \omega_{0,\iota}^{(m)} \upharpoonright \mathcal{A}_{0,\iota}^{(m)}(\mathcal{O}_a^{(+)}). \quad (4.32)$$

On the other hand, if \mathcal{O} is contained in $\mathcal{O}_a^{(-)}$ it follows from locality that

$$\langle V_q^{(n)} \Omega_{0,\iota}, \pi_{0,\iota}(\underline{A}) V_q^{(n)} \Omega_{0,\iota} \rangle = \langle \Omega_{0,\iota}, \pi_{0,\iota}(\underline{A}) \Omega_{0,\iota} \rangle. \quad (4.33)$$

Hence relation (4.32) holds also with $\mathcal{O}^{(+)}$ replaced by $\mathcal{O}^{(-)}$. Thus we have proved (b.i).

Let us next show that the state $\omega_{q,\iota}^{(m)}$ is disjoint from $\omega_{0,\iota}^{(m)}$ on $\mathcal{A}_{0,\iota}^{(m)}(\mathcal{O}')$ for every double cone \mathcal{O}' . We begin by noting that the unitaries $Z_w^{(n)}(r)$ are contained in the local von Neumann algebras $\mathcal{A}_{0,\iota}^{(m)}(\mathcal{O}_{n,w})^-$ for suitable double cones $\mathcal{O}_{n,w}$ and, by local normality, the state $\omega_{q,\iota}^{(m)}$ extends to these algebras. On the other hand, we have that $Z_w^{(n+1)}(r) \in \mathcal{A}_{0,\iota}^{(m)}(\mathcal{O}_{n,+})^- \cup \mathcal{A}_{0,\iota}^{(m)}(\mathcal{O}_{n,-})^-$ where $\mathcal{O}_{n,\pm}$ are double cones in the right/left spacelike complement of the double cone \mathcal{O}_n based on the interval $(-na, na)$. This follows from the support properties of the functions g_n and the Weyl-relations. Since $Z_w^{(n)}(r)$ forms a central sequence in $\mathcal{A}_{0,\iota}^{(m)}(\mathcal{O}')^-$, it is sufficient for the proof of the claimed disjointness to show that, for each $\varepsilon > 0$, there is some real-valued $w \in \mathcal{D}(\mathbb{R})$ with $w(\mathbf{x}) = 1$ for $|\mathbf{x}| < 1$, such that in the limit of large n ,

$$|\omega_{0,\iota}^{(m)}(Z_w^{(n)}(\pi/q)) - \omega_{q,\iota}^{(m)}(Z_w^{(n)}(\pi/q))| \geq 2 - \varepsilon. \quad (4.34)$$

That this can be achieved may be shown as follows. Let $\mathcal{C} := \{w \in \mathcal{D}(\mathbb{R}) : w \text{ real, } w(\mathbf{x}) = 1 \text{ for } |\mathbf{x}| < 1\}$, and let $\varepsilon > 0$ be given. We claim that there is some $w \in \mathcal{C}$ having the property

$$2 \cdot e^{-(\pi^2/4q^2) \cdot \int d\mathbf{k} |\mathbf{k}| |\tilde{w}(\mathbf{k})|^2} \geq 2 - \varepsilon. \quad (4.35)$$

It is plain that this amounts to showing that there holds

$$\inf_{w \in \mathcal{C}} \int d\mathbf{k} |\mathbf{k}| |\tilde{w}(\mathbf{k})|^2 = 0. \quad (4.36)$$

Now $w \in \mathcal{C}$ implies that also $w_j \in \mathcal{C}$, where

$$w_j(\mathbf{x}) := w(\mathbf{x}/j), \quad j \in \mathbb{N}. \quad (4.37)$$

On the other hand, $\int d\mathbf{k} |\mathbf{k}| |\tilde{w}_j(\mathbf{k})|^2 = \int d\mathbf{k} |\mathbf{k}| |\tilde{w}(\mathbf{k})|^2$ for all $j \in \mathbb{N}$, and for all $u \in \mathcal{D}(\mathbb{R})$ there holds in the limit of large j

$$\int d\mathbf{k} u(\mathbf{k}) |\mathbf{k}|^{1/2} \tilde{w}_j(\mathbf{k}) = \int d\mathbf{k} j^{-1/2} u(j^{-1}\mathbf{k}) |\mathbf{k}|^{1/2} \tilde{w}(\mathbf{k}) \rightarrow 0. \quad (4.38)$$

This shows that the sequence of functions $\mathbf{k} \mapsto |\mathbf{k}|^{1/2} \tilde{w}_j(\mathbf{k})$, $j \in \mathbb{N}$, tends weakly to 0 in $L^2(\mathbb{R})$ for $j \rightarrow \infty$. Since \mathcal{C} is a convex set there exists then another, averaged sequence $\overline{w}_j \in \mathcal{C}$ such that $\int d\mathbf{k} |\mathbf{k}| |\tilde{\overline{w}}_j(\mathbf{k})|^2 \rightarrow 0$ for $j \rightarrow \infty$. Thus we arrive at (4.36), and consequently there is for the prescribed $\varepsilon > 0$ a $w \in \mathcal{C}$ satisfying (4.35).

Our next claim is that, with such a $w \in \mathcal{C}$, relation (4.34) is satisfied for all $n \in \mathbb{N}$. To this end we recall that in view of the Weyl-relations and locality there is for any choice of $n \in \mathbb{N}$ some $j \in \mathbb{N}$ such that

$$\begin{aligned} \omega_{q,\iota}^{(m)}(Z_w^{(n)}(\pi/q)) &= \langle V_q^{(j)} \Omega_{0,\iota}, Z_w^{(n)}(\pi/q) V_q^{(j)} \Omega_{0,\iota} \rangle \\ &= e^{i(\pi/q) \int d\mathbf{x} u_j(\mathbf{x}) g_n(\mathbf{x})} \langle \Omega_{0,\iota}, Z_w^{(n)}(\pi/q) \Omega_{0,\iota} \rangle. \end{aligned} \quad (4.39)$$

For the integral in the exponential we obtain for sufficiently large j

$$\int d\mathbf{x} u_j(\mathbf{x}) g_n(\mathbf{x}) = \int d\mathbf{x} u_j(\mathbf{x}) \partial_{\mathbf{x}} w(\mathbf{x}/n) = -q. \quad (4.40)$$

This is a consequence of the support properties of the functions involved: $\partial_{\mathbf{x}} w(\mathbf{x}/n)$ is supported on two disjoint regions close to $\pm n$, and u_j vanishes on the region at $-n$ and equals q on the other one. Therefore, with the help of Lemma 4.3 and the preceding estimates, we arrive at

$$\begin{aligned} |\omega_{0,\iota}^{(m)}(Z_w^{(n)}(\pi/q)) - \omega_{q,\iota}^{(m)}(Z_w^{(n)}(\pi/q))| &= (1 - e^{-i\pi}) \langle \Omega_{0,\iota}, Z_w^{(n)}(\pi/q) \Omega_{0,\iota} \rangle \quad (4.41) \\ &= 2 \cdot e^{-(\pi^2/4q^2) \cdot \int d\mathbf{k} |\mathbf{k}| |\tilde{w}(\mathbf{k})|^2} \geq 2 - \varepsilon. \end{aligned}$$

This completes the proof of statement (b.ii) in the theorem.

We mention as an aside that by the same argument one sees that also the states $\omega_{q,\iota}^{(m)}$ corresponding to different charge values q are disjoint on the algebras $\mathcal{A}_{0,\iota}^{(m)}(\mathcal{O})$. Hence the charge of these states can be determined in the spacelike complement of any double cone, which may be interpreted as a manifestation of Gauss' law.

It remains to establish the last statement (b.iii) of the theorem concerning the implementation of the translations in the representations induced by $\omega_{q,\iota}^{(m)}$. We indicate here only the essential steps in the argument and refrain from giving the necessary computations as they are of a similar nature as in the preceding steps.

We begin by noting that the representation $\varrho_{q,\iota}$ of $\mathcal{A}_{0,\iota}^{(m)}$, which is fixed by $\omega_{q,\iota}^{(m)}$, can be realized on the Hilbert space $\mathcal{H}_{0,\iota}$ by setting

$$\varrho_{q,\iota}(\pi_{0,\iota}(\underline{A})) := \lim_n V_q^{(n)*} \pi_{0,\iota}(\underline{A}) V_q^{(n)}, \quad (4.42)$$

with $V_q^{(n)}$ as in (4.28). This limit exists in the norm topology because of the Weyl-relations and locality. We note in passing that $\varrho_{q,\iota}$ is an automorphism of the C^* -algebra generated by the local von Neumann algebras $\mathcal{A}_{0,\iota}^{(m)}(\mathcal{O})^-$.

The translations in the representation $\varrho_{q,\iota}$ are obtained by setting

$$U_{q,\iota}(x) := \lim_n V_q^{(n)*} U_{0,\iota}(x) V_q^{(n)}. \quad (4.43)$$

Here $U_{0,\iota}(x)$ are the unitary translation operators in the defining representation of the net $\mathcal{A}_{0,\iota}^{(m)}$ on $\mathcal{H}_{0,\iota}$. The somewhat laborious task is to show that this limit exists for a suitable choice of the functions u_n in the definition of $V_q^{(n)}$. In contrast to the preceding results where the specific form of u_n was, for $\mathbf{x} > na$, completely arbitrary (cf. (4.29)), the proof that the translations can be represented in the form (4.43) requires a proper choice of these functions in that region. With these preparations one can show that the operator $V_q^{(n)*} U_{0,\iota}(x) V_q^{(n)} U_{0,\iota}(x)^*$ can be decomposed into a product $Z_x^{(n)} \Gamma_x$ of Weyl-operators, where Γ_x does not depend on n and $Z_x^{(n)}$ is a central sequence whose vacuum expectation value converges to 1, uniformly on compact sets in x . Since each $Z_x^{(n)}$ is unitary, the uniform convergence of (4.43) in the strong-operator topology follows.

It is then clear that (4.43) defines a continuous unitary representation of the translations on $\mathcal{H}_{0,\iota}$ which satisfies the relativistic spectrum condition (since $U_{0,\iota}$

does). Combining relation (4.42) with (4.43) one also sees that the unitaries $U_{q,\iota}(x)$ implement the translations $\alpha_x^{(m;0,\iota)}$ in the representation $\varrho_{q,\iota}$ and this completes the proof of the theorem.

5 Some open problems

The present investigation of the short distance properties of free field theories has produced some interesting results which corroborate the general ideas expounded in [8]. Yet in spite of the basic simplicity of the underlying class of models there remain some intriguing questions whose understanding seems to be of importance for the treatment of less trivial examples, notably interacting theories.

First, there is the role of dimension. In our computation of the scaling limit of free field theories in Sec. 2 we relied heavily on the fact that the corresponding vacuum states are, in $s = 2$ and 3 dimensions, locally normal with respect to each other for any value $m \geq 0$ of the mass. In $s > 3$ dimensions this is no longer true because of ultraviolet problems and this fact resembles the situation which one expects to encounter in interacting theories in physical spacetime. There the scaled vacuum states (corresponding to different “running” values of the coupling constants and masses) are most likely locally disjoint. Hence it would be of interest to develop in the simpler case of free field theory in $s > 3$ dimensions techniques which allow one to compute the scaling limit nets $(\mathcal{A}_{0,\iota}^{(m)}, \alpha^{(m;0,\iota)})$ without relying on local normality. It is clear from our present arguments that also in these models the local algebras $\mathcal{A}_{0,\iota}^{(m)}(\mathcal{O})$ and $\mathcal{A}^{(0)}(\mathcal{O})$ have large sub-algebras in common which consist of smoothed-out Weyl-operators. But this information is not yet sufficient in order to clarify the relation between the respective nets.

Because of similar reasons we neither have an explicit description of the scaling limit theory in $s = 1$ dimensions, nor do we know whether this scaling limit is unique. If one restricts attention to the sub-algebra of the scaling algebra which is generated by smoothed-out Weyl-operators, then one can show that the resulting subnets of the scaling limit theory are isomorphic to the nets generated by smoothed-out Weyl-operators in massless free field theory, tensored with an Abelian algebra [3]. It is an open problem whether an analogous result holds if one starts from the full scaling algebra.

Within the realm of free field theory it would also be of interest to determine the scaling limit of the local net generated by a free massive vector field, which resembles in certain respects the Higgs model. It would be interesting to see whether in the free field case there exist in the scaling limit physical states which carry a charge that is screened at finite scales, similarly as in the Schwinger model.

The real challenge, however, is the short distance analysis of interacting theories with the method of the scaling algebra. There the simplest examples are the $\mathcal{P}(\phi)_2$ -models which are known to be asymptotically free (super-renormalizable) and to have vacuum states which are locally normal with respect to the massive Fock vacuum. Because of this close relationship to free field theory one may hope that the present results will also be of use in the analysis of these more interesting examples.

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